Topological realizations of CBERs

<u>Def:</u> $E \subseteq X \times X$ is a <u>CBER</u> if E is a Borel equiv rel on X and every E-class is ctble.

E is <u>aperiodic</u> if every E-class is infinite.

<u>Def:</u> Given CBERs E, F on X, Y, respectively, $E \cong_B F$ if $\exists$ Borel bij $\varphi : X \to Y$ s.t. $\forall x, x' \in X \quad x E x' \iff \varphi(x) F \varphi(x')$.

Rmk: Borel iso is stronger than Borel bireducibility

$$\boxed{E_0} \quad \text{vs.} \quad \boxed{E_0 \mid E_0}$$

Def: A topological realization of a CBER E is a CBER F on a Polish space Y s.t. $E \cong_B F$.

Q: What properties can a top realiz have?

Prop (Frisch-Kechris-Shinko-Vidnyánszky): Every aper CBER admits a top realiz induced by a cts action of a ctble group on $\mathbb{N}^{\mathbb{N}}$.

Compact action realiz

Q: When does an aper CBER admit a top realiz induced by a cts action of a ctble group on a compact Polish space?

Thm (FKSV):

1. If an aper CBER admits a compact action realiz, then it is not smooth.
2. Every non-smooth hyperfinite aper CBER has a minimal compact action realiz on $2^{\mathbb{N}}$.

3. Let Γ be an infinite ctble group, and consider the shift action $\Gamma \curvearrowright (2^{\mathbb{N}})^\Gamma$. The restriction of $E_\Gamma^{(2^{\mathbb{N}})^\Gamma}$ to the free part has a compact action realiz.

<u>K_σ realiz</u>

<u>Q (Conley)</u>: Does every aper CBER have a K_σ realiz?

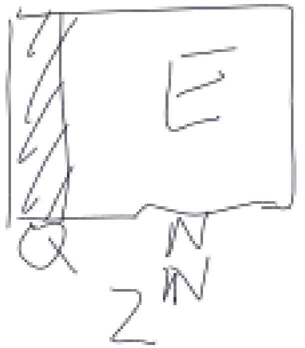
Observe: If X compact, Γ ctble, $\Gamma \curvearrowright X$ cts, then E_Γ^X is K_σ.

Thm (FKSV): Every aper CBER E admits a transitive K_σ realiz on $2^{\mathbb{N}}$.

Pf: Let $Q := \{x \in 2^{\mathbb{N}} \mid x \text{ has finitely many } 1\text{'s}\}$ and $N := 2^{\mathbb{N}} \setminus Q$.

Since N and $\mathbb{N}^{\mathbb{N}}$ are homeomorphic, we can assume there is ctble Γ and cts $\Gamma \curvearrowright N$ s.t. $E = E_\Gamma^N$.

Let $E' = E \cup (Q \times Q)$.



Fact: $E' \cong_B E$.

For $\gamma \in \Gamma$, let $R_\gamma = \{(x, \gamma \cdot x) \mid x \in N\}$, and let $\overline{R_\gamma}$ be the closure of R_γ in $2^{\mathbb{N}} \times 2^{\mathbb{N}}$.

<u>Clm:</u> If $(x,y) \in \overline{R_\gamma}$, then $(x,y) \in R_\gamma \cup (Q \times Q) \subseteq E'$.

Assuming the clm, $E' = \bigcup_{\gamma \in \Gamma} \overline{R_\gamma} \cup (Q \times Q)$ is K_σ.

<u>Pf of Clm:</u>

Suppose $(x,y) \in \overline{R_\gamma}$ and $x \in N$. We will show $y = \gamma \cdot x$.

Fix seq $(x_n, y_n)_{n \in \mathbb{N}}$ in R_γ s.t. $(x_n, y_n) \longrightarrow (x, y)$.

Since $x_n \longrightarrow x$ and γ acts ctsly, $\underbrace{\gamma \cdot x_n}_{= y_n} \longrightarrow \gamma \cdot x$. So $y = \gamma \cdot x$. $\square_{\text{Clm}}$

$\square_{\text{Thm}}$

Compactly graphable realiz

__Def:__ A CBER E on a compact Polish space X is __compactly graphable__ if $\exists$ compact graph $K \subseteq X \times X$ s.t. the E-classes are the K-connected components.

Observation: Every compactly graphable CBER is K_σ.

__Thm (FKSV):__ Every aper CBER that is hyperfinite or compressible admits a compactly graphable realiz.

__Prop (W):__ Suppose Γ is ctble, $\Gamma \curvearrowright 2^{\mathbb{N}}$ cts, and $E = E_\Gamma^{2^{\mathbb{N}}}$ is aper. If E is minimal and some $\gamma \in \Gamma$ acts fixed-pt-freely, then E is compactly graphable.